

A class of exactly solvable rationally extended Calogero-Wolfes type 3-body problems

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Abstract

In this work, we start from the well known Calogero-Wolfes type 3-body problems on a line and construct the corresponding exactly solvable rationally extended 3-body potentials. In particular, we obtain the corresponding energy eigenvalues and eigenfunctions which are in terms of the product of X_m Laguerre and X_p Jacobi exceptional orthogonal polynomials where both $m, p = 1, 2, 3, \dots$

1 Introduction

There are very few exactly solvable quantum many body problems even in one dimension. One of the first successful attempt was made by Calogero [1] in 1969 who gave exact solution for the problem of three particles interacting pairwise by inverse cube forces (i.e. inverse square potential) in addition to linear forces (i.e. harmonic potential). Thereafter Calogero and Sutherland obtained the solutions of more general N -body problems in one dimension [2, 3]. Other one dimensional many body problems of the Calogero and the Sutherland types and their applications are also discussed in Refs. [4, 5, 6, 7, 8, 9].

In 1974 Wolfes [10] showed that the Calogero approach could be generalized and one can obtain exact solution of new three-body problems in one dimension. Later on Khare and Bhaduri [11] extended the Calogero-Wolfes approach and obtained exact solutions of a number of three-body potentials in one dimension by using the ideas of supersymmetric quantum mechanics [12]. After the development of PT (combined parity and time reversal) symmetric quantum mechanics [13], people also considered the complex but PT -invariant extension of the Calogero model and showed that in this case too one has real bound state energy eigenvalue spectrum [14, 15, 16, 17].

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It may be noted that the corresponding bound state eigenfunctions are associated with the well known classical orthogonal polynomials.

After the recent discovery of two new orthogonal polynomials namely the exceptional X_1 Laguerre and X_1 Jacobi (or more general X_m Laguerre and X_p Jacobi respectively) polynomials [18, 19, 20, 21], a number of one body exactly solvable potentials [22, 23, 24, 25, 26, 27, 28, 29, 30] as well as Calogero type many particle potentials [31] have been extended rationally whose solutions are obtained in terms of these exceptional orthogonal polynomials (EOPs). Many interesting properties of these extended potentials have been studied [32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. It is then worth enquiring if one can construct new Calogero-Wolfes type three-body problems whose exact solutions are given in terms of these EOPs. This is the task that we have addressed in this paper.

In particular, in this paper, we have added to the list of exactly solvable three-body problems in one dimension by constructing a number of rationally extended three-body problems whose eigenfunctions are in terms of exceptional Laguerre as well as Jacobi polynomials (EOPs). Remarkably, we have also been able to construct complex rationally extended PT -invariant three-body problems whose solutions are not in the exact forms of EOPs rather they are written in the forms of some new type of polynomials which can be further expanded in terms of classical Jacobi polynomials.

The plan of the paper is as follows: In section 2, we briefly recall the Calogero type as well as Wolfes type three body problems and how the choice of Jacobi coordinates play a crucial role in obtaining the exact solution of such three-body problems. In section 3, we explain our ideas by extending the Calogero-Wolfes type three body problems by adding new three-body interaction terms and obtain their solution in terms of the product of the X_1 exceptional Laguerre and X_1 exceptional Jacobi polynomials. The generalization to the X_m and X_p polynomials in Laguerre and Jacobi variables respectively is also discussed. In subsection 3.1, we discuss another example where we add complex but PT -invariant three-body interaction terms and show that the solution in general is product of X_m Laguerre polynomials times some new polynomial which can be expanded in terms of classical Jacobi orthogonal polynomials.

A list of all possible rationally extended real and PT symmetric complex three body problems whose solution is in terms of the product of the X_1 Laguerre and X_1 Jacobi polynomials is given in Table 1. A similar list of all possible rationally extended real and PT symmetric complex three body problems whose solution is in terms of the product of the X_m Laguerre and X_p Jacobi polynomials is given in Table 2. Finally, we summarize our results in section 4.

2 The Calogero-Wolfes type three body problems

The Calogero's [1] exactly solvable three body problem is characterized by the potential

$$V_C = V_H + V_I, \quad (1)$$

where V_H and V_I are harmonic and inverse square potentials

$$V_H = \frac{\omega^2}{8} \sum_{i < j} (x_i - x_j)^2; \quad V_I = g \sum_{i < j} (x_i - x_j)^{-2}, \quad (2)$$

respectively. Here $g > -1/2$ is a coupling parameter used to avoid a collapse of the system. This three body problem is solved exactly by defining the Jacobi co-ordinates

$$R = \frac{1}{3}(x_1 + x_2 + x_3),$$

and

$$x = \frac{(x_1 - x_2)}{\sqrt{2}}, \quad y = \frac{(x_1 + x_2 - 2x_3)}{\sqrt{6}}. \quad (3)$$

In polar co-ordinates

$$x = r \sin \phi, \quad y = r \cos \phi; \quad 0 \leq r \leq \infty, \quad 0 \leq \phi \leq 2\pi, \quad (4)$$

with

$$r^2 = \frac{1}{3}[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2]; \quad x_1 \neq x_2 \neq x_3. \quad (5)$$

Using Eqs. (3) and (4), one can easily show that

$$\begin{aligned} (x_1 - x_2) &= \sqrt{2}r \sin \phi, \\ (x_2 - x_3) &= \sqrt{2}r \sin(\phi + 2\pi/3), \\ (x_3 - x_1) &= \sqrt{2}r \sin(\phi + 4\pi/3). \end{aligned} \quad (6)$$

In 1974 Wolfes [10] showed that a three-body potential

$$V_W(g) = g[(x_1 + x_2 - 2x_3)^{-2} + (x_2 + x_3 - 2x_1)^{-2} + (x_3 + x_1 - 2x_2)^{-2}] \quad (7)$$

is also solvable when it is added to V_C with or without the inverse square potential V_I . Note that the last two terms in the potential (7) are the cyclic permutation of the first one. Hence from now onwards, we will refer to such terms as c.p. i.e. (cyclic permutation terms).

3 Rationally extended three body potentials

In this section, we follow the method adopted in [11] and obtain some new three body potentials whose solutions are in terms of EOPs. As an illustration, we first consider a potential of the form

$$V = V_H + V_W(g) + V_{int} + V_{rat}, \quad (8)$$

where V_H and $V_W(g)$ are as given by Eqs. (2) and (7) respectively while V_{int} is given by

$$V_{int} = \frac{3f_1}{2\sqrt{2}r} \left[\frac{(x_1 - x_2)}{(x_1 + x_2 - 2x_3)^2} + \text{c.p.} \right]. \quad (9)$$

Finally, the newly added rational term V_{rat} is defined as

$$V_{rat} = V_{rat}^{(1)} + V_{rat}^{(2)}, \quad (10)$$

where $V_{rat}^{(1)}$, $V_{rat}^{(2)}$ are the two new rational interaction terms introduced by us. Thus the combined potential becomes

$$V = \frac{\omega^2}{8} \sum_{i < j} (x_i - x_j)^2 + 3g[(x_1 + x_2 - 2x_3)^{-2} + \text{c.p}] + \frac{3f_1}{2\sqrt{2}r} \left[\frac{(x_1 - x_2)}{(x_1 + x_2 - 2x_3)^2} + \text{c.p} \right] + V_{rat}^{(1)} + V_{rat}^{(2)}. \quad (11)$$

Now we define $V_{rat}^{(1)}$ and $V_{rat}^{(2)}$ as

$$V_{rat}^{(1)} = \frac{a \sum_{i < j} (x_i - x_j)^2 + c_1}{(b \sum_{i < j} (x_i - x_j)^2 + c_2)^2} \quad (12)$$

and

$$V_{rat}^{(2)} = \frac{\delta}{r^2} \left[\frac{k_1}{(k_2 + k_3\xi)} - \frac{k_4}{(k_2 + k_3\xi)^2} \right]. \quad (13)$$

Here a , b , c_1 , c_2 , δ , k_1 , k_2 , k_3 , k_4 are constants while ξ is a function of x_1, x_2, x_3 . The values of these constants and the form of the function ξ depend on the nature of the problems. For example, if the function ξ is of the forms

$$\xi = \frac{4}{2\sqrt{2}r^3} (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \quad (14)$$

or

$$\xi = -\frac{3}{\sqrt{2}r} \left(\sum_{i < j} (x_i - x_j)^{-1} \right)^{-1}, \quad (15)$$

it leads to two different three body problems. On using the Jacobi co-ordinates (3)-(6) and the following identities

$$\begin{aligned} \prod_{s=1}^3 \sin\left(\phi + \frac{2(s-1)\pi}{3}\right) &= -\frac{1}{4} \sin(3\phi), \\ \sum_{s=1}^3 \text{cosec}\left(\phi + \frac{2(s-1)\pi}{3}\right) &= 3 \text{cosec}(3\phi), \end{aligned} \quad (16)$$

remarkably, both the forms of the above ξ , reduce to a same function

$$\xi = -\sin(3\phi). \quad (17)$$

In polar co-ordinates (r, ϕ) , the Schrödinger equation corresponding to the above potential is given by ($\hbar = 2m = 1$)

$$\left[-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \frac{d^2}{d\phi^2} \right] \psi_{n\ell}(r, \phi) + V \psi_{n\ell}(r, \phi) = E_{n\ell} \psi_{n\ell}(r, \phi). \quad (18)$$

Note that the potential is non-central but separable and hence one can write the wave function in the form

$$\psi_{n\ell}(r, \phi) = \frac{R_{n\ell}(r)}{r^{1/2}} \Phi_{\ell}(\phi). \quad (19)$$

It is then straightforward to show that the radial component of the wave function satisfies the equation

$$\left[-\frac{d^2}{dr^2} + V(r) \right] R_{n\ell}(r) = E_{n\ell} R_{n\ell}(r), \quad (20)$$

where the potential $V(r)$ is given by

$$V(r) = V_{Con}(r) + V_{rat}^{(1)}(r). \quad (21)$$

Here $V_{Con}(r)$ is the conventional radial oscillator potential, i.e.

$$V_{Con}(r) = \frac{3}{8}\omega^2 r^2 + \frac{(\lambda_\ell^2 - 1/4)}{r^2}, \quad (22)$$

while the rational term $V_{rat}^{(1)}(r)$ is given by

$$V_{rat}^{(1)}(r) = \frac{(3ar^2 + c_1)}{(3br^2 + c_2)^2}. \quad (23)$$

Now notice that in case we set the parameters a , b , c_1 and c_2 as

$$a = 2\omega^2; \quad 3b = (\sqrt{3/2})\omega; \quad c_1 = -4(\sqrt{3/2})\omega c_2 \quad \text{and} \quad c_2 = 2\lambda_\ell, \quad (24)$$

then the above potential (21) matches exactly with the rationally extended radial oscillator potential given in [22]. As shown there, in that case, the bound states eigenfunctions are given by the X_1 Laguerre polynomials

$$R_{n\ell}(r) \propto \frac{r^{\lambda_\ell+1/2} \exp\left(-\frac{1}{4}(\sqrt{3/2})\omega r^2\right) \hat{L}_{n+1}^{(\lambda_\ell)}((\sqrt{3/8})\omega r^2)}{[(\sqrt{3/2})\omega r^2 + 2\lambda_\ell]}, \quad (25)$$

and the corresponding energy eigenvalues are

$$E_{n\ell} = (\sqrt{3/2})\omega(2n + \lambda_\ell + 1); \quad n = 0, 1, 2, \dots; \quad \ell = 0, 1, 2, \dots, \quad (26)$$

with $\lambda_\ell > 0$.

On the other hand, the angular part of the eigenfunction $\Phi_\ell(\phi)$ can be shown to satisfy the Schrödinger equation

$$\left[-\frac{d^2}{d\phi^2} + V(\phi) \right] \Phi_\ell(\phi) = \lambda_\ell^2 \Phi_\ell(\phi), \quad (27)$$

where

$$\begin{aligned} V(\phi) &= \frac{g}{2} \sum_{s=1}^3 \sec^2\left(\phi + \frac{2(s-1)\pi}{3}\right) + \frac{f_1}{2} \sum_{s=1}^3 \sec\left(\phi + \frac{2(s-1)\pi}{3}\right) \\ &\times \tan\left(\phi + \frac{2(s-1)\pi}{3}\right) + V_{rat}^{(2)}(\phi). \end{aligned} \quad (28)$$

Using the following identities

$$\sum_{s=1}^3 \sec^2\left(\phi + \frac{2(s-1)\pi}{3}\right) = 9 \sec^2(3\phi), \quad (29)$$

and

$$\sum_{s=1}^3 \sec\left(\phi + \frac{2(s-1)\pi}{3}\right) \tan\left(\phi + \frac{2(s-1)\pi}{3}\right) = -9 \tan(3\phi) \sec(3\phi), \quad (30)$$

the potential (28) takes the form

$$V(\phi) = V_{Con}(\phi) + V_{rat}^{(2)}(\phi), \quad (31)$$

where $V_{Con}(\phi)$ is essentially the conventional trigonometric Scarf potential

$$V_{Con}(\phi) = \frac{9g}{2} \sec^2(3\phi) - \frac{9f_1}{2} \sec(3\phi) \tan(3\phi), \quad (32)$$

while the rational term $V_{rat}^{(2)}(\phi)$ takes the form

$$V_{rat}^{(2)}(\phi) = \delta \left[\frac{k_1}{(k_2 - k_3 \sin(3\phi))} - \frac{k_4}{(k_2 - k_3 \sin(3\phi))^2} \right]. \quad (33)$$

On comparing the above obtained potential (31) with the rationally extended trigonometric Scarf potential as given in [22], i.e.

$$\begin{aligned} V(\phi) &= [A(A-3) + B^2] \sec^2(3\phi) - B(2A-3) \sec(3\phi) \tan(3\phi) \\ &+ 9 \left[\frac{2(2A-3)}{(2A-3-2B \sin(3\phi))} - \frac{2[(2A-3)^2 - 4B^2]}{(2A-3-2B \sin(3\phi))^2} \right], \end{aligned} \quad (34)$$

defined over the range $-\frac{\pi}{6} < \phi < \frac{\pi}{6}$ and the parametric restriction $0 < B < A-3$, we get

$$\begin{aligned} A(A-3) + B^2 &= \frac{9g}{2}; \\ B(2A-3) &= \frac{9f_1}{2}, \end{aligned} \quad (35)$$

$$\delta = 9; \quad k_1 = 2k_2; \quad k_2 = 2A-3; \quad k_3 = 2B; \quad \text{and} \quad k_4 = 2(k_2^2 - k_3^2). \quad (36)$$

To get explicit expressions for A and B in terms of g and f_1 , we solve Eq. (35) and obtain four possible roots for A and B . Out of all these, we consider one suitable root given by

$$\begin{aligned} A &= \frac{1}{16} [24 + 12\sqrt{2(1+2g+\zeta)}]; \quad B = \frac{3f_1}{\sqrt{2(1+2g+\zeta)}}, \\ \text{where } \zeta &= \sqrt{(1+2g)^2 - 4f_1^2}. \end{aligned} \quad (37)$$

The bound state wave functions of $V(\phi)$ are well known and given in terms of X_1 exceptional Jacobi polynomial $\hat{L}_{\ell+1}^{(\alpha,\beta)}(\sin(3\phi))$ as

$$\Phi_\ell(\phi) \propto \frac{(1 - \sin(3\phi))^{\frac{1}{6}(A-B)} (1 + \sin(3\phi))^{\frac{1}{6}(A+B)}}{(2A-3-2B \sin(3\phi))} \hat{P}_{\ell+1}^{(\alpha,\beta)}(\sin(3\phi)), \quad (38)$$

with the parameters

$$\begin{aligned} \alpha &= \left(\frac{A}{3} - \frac{B}{3} - \frac{1}{2} \right) = \frac{1}{2} \sqrt{1+2(g-f_1)}; \\ \beta &= \left(\frac{A}{3} + \frac{B}{3} - \frac{1}{2} \right) = \frac{1}{2} \sqrt{1+2(g+f_1)}, \end{aligned} \quad (39)$$

and the energy eigenvalues

$$\lambda_\ell^2 = (A + 3\ell)^2; \quad \ell = 0, 1, 2, \dots, \quad (40)$$

Summarizing, for the three-body problem with the potentials as given by Eq. (8), the exact energy eigenvalues are given by Eq. (26) where using the expression for A as given by Eq. (37), λ_ℓ as given by Eq. (40) takes the form

$$\lambda_\ell = \frac{3}{4}[2(2\ell + 1) + \sqrt{2(1 + 2g + \zeta)}], \quad (41)$$

with ζ being given by Eq. (37). Note that ζ and hence λ_ℓ are real only if $f_1 < g + 1/2$. The corresponding eigenfunctions are given by Eq. (19) where $R_\ell(r)$ and $\Phi_\ell(\phi)$ are as given by Eqs. (25) and (38) respectively.

Generalization to the X_m case:

The above three body problems can be now easily extended to a more general three-body problem where the solutions of both the radial and the angular parts is in terms of X_m Laguerre and X_p Jacobi polynomials by redefining Eqs. (12) and (13) as

$$\begin{aligned} V_{rat}^{(1)} \Rightarrow V_{rat}^{(1)}(m) &= -2m(\sqrt{3/2})\omega - \frac{\omega^2}{2} \sum_{i < j} (x_i - x_j)^2 \frac{L_{m-2}^{(\lambda_\ell+1)}(-(\sqrt{3/8})\omega r^2)}{L_m^{(\lambda_\ell-1)}(-(\sqrt{3/8})\omega r^2)} \\ &+ (\sqrt{3/2})\omega \left[(\sqrt{1/6})\omega \sum_{i < j} (x_i - x_j)^2 + 2\lambda_\ell - 2 \right] \frac{L_{m-1}^{(\lambda_\ell)}(-(\sqrt{3/8})\omega r^2)}{L_m^{(\lambda_\ell-1)}(-(\sqrt{3/8})\omega r^2)} \\ &+ \omega^2 \sum_{i < j} (x_i - x_j)^2 \left(\frac{L_{m-1}^{(\lambda_\ell)}(-(\sqrt{3/8})\omega r^2)}{L_m^{(\lambda_\ell-1)}(-(\sqrt{3/8})\omega r^2)} \right)^2, \end{aligned} \quad (42)$$

and

$$\begin{aligned} V_{rat}^{(2)} \Rightarrow V_{rat}^{(2)}(p) &= \frac{\delta}{r^2} \left[-2p(\alpha - \beta - p + 1) - (\alpha - \beta - p + 1)(\alpha + \beta + (\alpha - \beta + 1)z) \right. \\ &\times \left. \frac{P_{p-1}^{(-\alpha, \beta)}(z)}{P_p^{(-\alpha-1, \beta-1)}(z)} + \frac{(\alpha - \beta - p + 1)^2 \chi}{2} \left(\frac{P_{p-1}^{(-\alpha, \beta)}(z)}{P_p^{(-\alpha-1, \beta-1)}(z)} \right)^2 \right], \end{aligned} \quad (43)$$

where

$$z = -\xi \quad \text{and} \quad \chi = 1 - z^2, \quad (44)$$

with a constant $\delta = 9$. Similar to the X_1 cases, using the concept of Jacobi co-ordinates, we get the radial and angular dependent potentials

$$\begin{aligned} V_{rat}^{(1)}(r) \Rightarrow V_{rat}^{(1)}(m, r) &= -2m(\sqrt{3/2})\omega - \frac{3\omega^2 r^2}{2} \frac{L_{m-2}^{(\lambda_\ell+1)}(-(\sqrt{3/8})\omega r^2)}{L_m^{(\lambda_\ell-1)}(-(\sqrt{3/8})\omega r^2)} \\ &+ (\sqrt{3/2})\omega \left[(\sqrt{3/2})\omega r^2 + 2\lambda_\ell - 2 \right] \frac{L_{m-1}^{(\lambda_\ell)}(-(\sqrt{3/8})\omega r^2)}{L_m^{(\lambda_\ell-1)}(-(\sqrt{3/8})\omega r^2)} \\ &+ 3\omega^2 r^2 \left(\frac{L_{m-1}^{(\lambda_\ell)}(-(\sqrt{3/8})\omega r^2)}{L_m^{(\lambda_\ell-1)}(-(\sqrt{3/8})\omega r^2)} \right)^2, \end{aligned} \quad (45)$$

and

$$\begin{aligned}
V_{rat}^{(2)}(\phi) \Rightarrow V_{rat}^{(2)}(p, \phi) &= 9 \left[-2p(\alpha - \beta - p + 1) - (\alpha - \beta - p + 1) \right. \\
&\times (\alpha + \beta + (\alpha - \beta + 1) \sin(3\phi)) \frac{P_{p-1}^{(-\alpha, \beta)}(\sin(3\phi))}{P_p^{(-\alpha-1, \beta-1)}(\sin(3\phi))} \\
&\left. + \frac{(\alpha - \beta - p + 1)^2 \cos^2(3\phi)}{2} \left(\frac{P_{p-1}^{(-\alpha, \beta)}(\sin(3\phi))}{P_p^{(-\alpha-1, \beta-1)}(\sin(3\phi))} \right)^2 \right],
\end{aligned} \tag{46}$$

respectively. The solutions of the Schrödinger equation for these potentials $V(r) \Rightarrow V(r, p)$ and $V(\phi) \Rightarrow V(p, \phi)$ in terms of X_m Laguerre and X_p Jacobi EOPs respectively are given as

$$R_{n\ell}(m, r) \propto \frac{r^{\lambda_\ell+1/2} \exp\left(-\frac{1}{4}(\sqrt{3/2})\omega r^2\right)}{L_m^{(\lambda_\ell-1)}(-(\sqrt{3/8})\omega r^2)} \hat{L}_{n+m}^{\lambda_\ell}((\sqrt{3/8})\omega r^2), \tag{47}$$

and

$$\Phi_\ell(p, \phi) \propto \frac{(1 - \sin(3\phi))^{\frac{1}{6}(A-B)} (1 + \sin(3\phi))^{\frac{1}{6}(A+B)}}{P_p^{(-\alpha-1, \beta-1)}(\sin(3\phi))} \hat{P}_{\ell+p}^{(\alpha, \beta)}(\sin(3\phi)). \tag{48}$$

The energy eigenvalues will be same as given in Eqs. (26), (40) and Eq. (41)..

By redefining the constants δ , k_1 , k_2 , k_3 , k_4 and the function ξ which is written in terms of x_1, x_2 and x_3 , two more three-body exactly solvable rationally extended real problems with their corresponding mapping potentials the RE trigonometric Pöschl-Teller II and RE trigonometric Pöschl-Teller potentials respectively are constructed whose solutions are in terms of the products of the exceptional X_1 Laguerre and X_1 Jacobi EOPs. The details are mentioned in Table 1. In the same way by redefining one more function i.e. χ , these two real potentials are further generalized to the product of X_m Laguerre and X_p Jacobi EOPs. The details of these are given in Table 2.

3.1 Rationally extended three-body PT symmetric complex potentials

In this section, we discuss some interesting rationally extended three body complex potentials whose mapping potentials are not real but they are complex and PT symmetric. The solutions of these potentials are not in the exact form of EOPs rather they are written in the form of some types of new polynomials discussed in Ref. [42] which are further expanded in terms of classical Jacobi polynomials.

Let us consider a complex potential V of the form

$$V = V_H + V_I + V_{int} + V_{rat}, \tag{49}$$

where V_H and V_I are given by Eq. (2), while V_{int} is given by

$$V_{int} = \frac{\sqrt{3}}{2r^2} i f_1 \left[\frac{(x_1 + x_2 - 2x_3)}{(x_1 - x_2)} + \text{c.p.} \right]. \tag{50}$$

Using $V_{rat} = V_{rat}^{(1)} + V_{rat}^{(2)}$, the combined potential becomes

$$V = \frac{\omega^2}{8} \sum_{i < j} (x_i - x_j)^2 + g \sum_{i < j} (x_i - x_j)^{-2} + \frac{\sqrt{3}}{2r^2} i f_1 \left[\frac{(x_1 + x_2 - 2x_3)}{(x_1 - x_2)} + \text{c.p.} \right] + V_{rat}^{(1)} + V_{rat}^{(2)}. \quad (51)$$

Here $V_{rat}^{(1)}$ is again as given in Eq. (12), however the form of $V_{rat}^{(2)}$ depends on the nature of the function ξ . If we define

$$\xi = \frac{1}{3\sqrt{3}} \left(\frac{x_1 + x_2 - 2x_3}{x_1 - x_2} + \text{c.p.} \right), \quad (52)$$

and using the identities

$$\sum_{s=1}^3 \text{cosec}^2 \left(\phi + \frac{2(s-1)\pi}{3} \right) = 9 \text{cosec}^2(3\phi) \quad (53)$$

and

$$\sum_{s=1}^3 \cot \left(\phi + \frac{2(s-1)\pi}{3} \right) = 3 \cot(3\phi), \quad (54)$$

then the ϕ -dependent potential will be

$$V(\phi) = V_{Con}(\phi) + V_{rat}^{(2)}(\phi), \quad (55)$$

with the equivalent conventional PT symmetric trigonometric Eckart potential

$$V_{Con}(\phi) = \frac{9}{2} g \text{cosec}^2(3\phi) + \frac{9}{2} i f_1 \cot(3\phi), \quad (56)$$

and the rational term

$$V_{rat}^{(2)}(\phi) = \delta \left[\frac{k_1}{(k_2 + k_3 \cot(3\phi))} + \frac{k_4}{(k_2 + k_3 \cot(3\phi))^2} \right]. \quad (57)$$

This potential is equivalent to the rationally extended PT symmetric complex trigonometric Eckart potential¹

$$V(\phi) = A(A-3) \text{cosec}^2(3\phi) + 2iB \cot(3\phi) + \frac{9}{A^2(A-3)^2} \left[\frac{-4iB[A^2(A-3)^2 - B^2]}{(iB + A(A-3) \cot(3\phi))} + \frac{2[A^2(A-3)^2 - B^2]^2}{(iB + A(A-3) \cot(3\phi))^2} \right], \quad (58)$$

with the constants

$$\begin{aligned} \delta = 9; \quad k_1 &= \frac{-4iB[A^2(A-3)^2 - B^2]}{A^2(A-3)^2}; \quad k_2 = iB; \quad k_3 = A(A-3); \\ \text{and } k_4 &= \frac{2[A^2(A-3)^2 - B^2]^2}{A^2(A-3)^2}. \end{aligned} \quad (59)$$

¹Which is easily obtained by complex co-ordinate transformation $x \rightarrow ix$ of the rationally extended hyperbolic Eckart potential given in [42].

The potential parameters A and B in terms of g and f_1 are related as

$$\begin{aligned} A &= \frac{3}{2} + 3a; \quad B = \frac{9}{4}f_1, \\ \text{where } a &= \frac{1}{2}\sqrt{1+2g}. \end{aligned} \quad (60)$$

By P (i.e. parity) we mean here $\phi \rightarrow \phi + \pi$ while by T (i.e. time reversal) we mean $t \rightarrow -t$ and $i \rightarrow -i$. The wavefunction associated with the above potential is product of the X_1 Laguerre polynomial as given by Eq. (25) times $\Phi_\ell(\phi)$ which is given by

$$\Phi_\ell(\phi) \propto \frac{(z-1)^{\frac{\alpha_\ell}{2}}(z+1)^{\frac{\beta_\ell}{2}}}{(iB + A(A-3)\cot(3\phi))} y_\ell^{(A/3, B/3)}(z), \quad (61)$$

with $z = i\xi = i\cot(3\phi)$. Here the polynomial function $y_\ell^{(A/3, B/3)}(z)$ can be expressed in terms of the classical Jacobi polynomials $P_\ell^{(\alpha_\ell, \beta_\ell)}(z)$ as

$$\begin{aligned} y_\ell^{(A/3, B/3)}(z) &= \frac{2(\ell + \alpha_\ell)(\ell + \beta_\ell)}{(2\ell + \alpha_\ell + \beta_\ell)} q_1^{(A/3, B/3)}(z) P_{\ell-1}^{(\alpha_\ell, \beta_\ell)}(z) \\ &- \frac{2(1 + \alpha_1)(1 + \beta_1)}{(2 + \alpha_1 + \beta_1)} P_\ell^{(\alpha_\ell, \beta_\ell)}(z). \end{aligned} \quad (62)$$

The parameters α_ℓ and β_ℓ in terms of A and B are given by

$$\alpha_\ell = -(A/3 - 1 + \ell) + \frac{B/9}{(A/3 - 1 + \ell)}; \quad \beta_\ell = -(A/3 - 1 + \ell) - \frac{B/9}{(A/3 - 1 + \ell)}. \quad (63)$$

and $q_1^{(A/3, B/3)}(z) = P_1^{(\alpha_1, \beta_1)}(z)$ (Classical Jacobi polynomial for $\ell = 1$). The energy eigenvalues are given by Eq. (26) where λ_ℓ is given by

$$\lambda_\ell^2 = 9(\ell + a - \frac{1}{2}) - \frac{9f_1^2}{16(\ell + a - \frac{1}{2})^2}; \quad \ell = 0, 1, 2, \dots \quad (64)$$

Note Φ_ℓ has to be in such order that $\lambda_\ell^2 > 0$, $f_1 < 4(\ell + a - 1/2)$.

Similar to the EOPs cases, the above complex potential can be generalized for any non-zero positive values of p by again considering the radial potential $V_{rat}^{(1)}(m)$ as given by Eq. (45) and defining $V_{rat}^{(2)}(p)$ by

$$\begin{aligned} V_{rat}^{(2)} \Rightarrow V_{rat}^{(2)}(p) &= \delta \sum_{i < j} (x_i - x_j)^{-2} \left[2i\xi \frac{\dot{q}_p^{(A/3, B/3)}(z)}{q_p^{(A/3, B/3)}(z)} - \frac{8r^2}{9} \sum_{i < j} (x_i - x_j)^{-2} \right. \\ &\times \left. \left(\frac{\ddot{q}_p^{(A/3, B/3)}(z)}{q_p^{(A/3, B/3)}(z)} - \left(\frac{\dot{q}_p^{(A/3, B/3)}(z)}{q_p^{(A/3, B/3)}(z)} \right)^2 \right) - p \right]. \end{aligned} \quad (65)$$

For $\delta = 16$ and using Jacobi co-ordinates, we get

$$\begin{aligned} V_{rat}^{(2)} \Rightarrow V_{rat}^{(2)}(p, \phi) &= -18\text{cosec}^2(3\phi) \left[2i \cot(3\phi) \frac{\dot{q}_p^{(A/3, B/3)}(z)}{q_p^{(A/3, B/3)}(z)} - \text{cosec}^2(3\phi) \right. \\ &\times \left. \left(\frac{\ddot{q}_p^{(A/3, B/3)}(z)}{q_p^{(A/3, B/3)}(z)} - \left(\frac{\dot{q}_p^{(A/3, B/3)}(z)}{q_p^{(A/3, B/3)}(z)} \right)^2 \right) - p \right]. \end{aligned} \quad (66)$$

The wavefunctions associated with this potentials will be product of X_m Laguerre times $\Phi_\ell(p, \phi)$ which is given by

$$\Phi_\ell(p, \phi) \propto \frac{(z-1)^{\frac{\alpha_\ell}{2}}(z+1)^{\frac{\beta_\ell}{2}}}{q_m^{(A/3, B/3)}(z)} y_{\nu, p}^{(A/3, B/3)}(z); \quad \nu = \ell + p - 1, \quad (67)$$

where $q_p^{(A/3, B/3)}(z) = P_p^{(\alpha_p, \beta_p)}$ and the polynomial function $y_{\nu, p}^{(A/3, B/3)}(z)$ is

$$\begin{aligned} y_{\nu, p}^{(A/3, B/3)}(z) &= \frac{2(\ell + \alpha_\ell)(\ell + \beta_\ell)}{(2\ell + \alpha_\ell + \beta_\ell)} q_p^{(A/3, B/3)}(z) P_{\ell-1}^{(\alpha_\ell, \beta_\ell)}(z) \\ &- \frac{2(p + \alpha_p)(p + \beta_p)}{(2p + \alpha_p + \beta_p)} q_{p-1}^{(A/3+1, B/3)}(z) P_\ell^{(\alpha_\ell, \beta_\ell)}(z), \end{aligned} \quad (68)$$

with the parameters

$$\alpha_p = -(A/3 - 1 + p) + \frac{B/9}{(A/3 - 1 + p)}; \quad \beta_p = -(A/3 - 1 + p) - \frac{B/9}{(A/3 - 1 + p)}. \quad (69)$$

The energy eigenvalues are again given by Eq. (26) where λ_l is given by Eq. (64).

Similar to the real cases, another 3-body complex problem with the PT symmetric complex mapping potential i.e the RE PT symmetric complex trigonometric Rosen-Morse potential can also be constructed by redefining all constant parameters and the function ξ . The details about this potential for $p = 1$ and then for any arbitrary values of p along with the other relevant parameters are given in detail in tables 1 and 2 respectively.

4 Summary and discussion

In this paper we have constructed several exactly solvable, rationally extended, Calogero-Wolfes type three body problems by adding new types of interaction terms. In particular, we have considered three extended problems and using the Jacobi co-ordinates, these extended problems have been transformed into the equivalent forms of RE exactly solvable real as well as PT symmetric complex potentials. We have constructed new potentials whose solutions are given as the product of the X_1 Laguerre times X_1 Jacobi polynomials. We have further generalized these potentials such that there solution is given in terms of the product of X_m Laguerre times X_p Jacobi polynomials ($m, p = 1, 2, 3, \dots$). While some of these potentials are real, others are complex but PT -symmetric potentials.

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Forms of potential (V)	Function ξ and other constant parameters	$V(\phi) = V_{Con}(\phi) + V_{rat}^{(2)}(\phi)$	λ_ℓ^2	$\Phi_\ell(\phi)$
(I) $V = V_H + V_W(g) + V_{int} + V_{rat}$, $V_W(g) = 3g[(x_1 + x_2 - 2x_3)^{-2} + c.p]$, $V_{int} = f_1 \sum (x_i - x_j)^{-2}$.	$\xi = \frac{9}{6r^2}[(x_1 + x_2 - 2x_3)^{-1} + c.t]^{-2}$ or $\xi = (\frac{4}{6\sqrt{6}})^2 \frac{1}{r^6} [(x_1 + x_2 - 2x_3) \times (x_2 + x_3 - 2x_1)(x_1 + x_3 - 2x_2)]^2$, $\delta = 36; k_1 = 2(\beta + \alpha)$; $k_2 = 2\beta; k_3 = -2(\beta - \alpha)$ $k_4 = 2[(\beta + \alpha)^2 - (\beta - \alpha)^2]$	$V_{Con}(\phi) = \frac{9g}{2} \sec^2(3\phi)$ $+ \frac{9f_1}{2} \operatorname{cosec}^2(3\phi)$; $V_{rat}^{(2)}(\phi) = 36[\frac{2(\beta+\alpha)}{(2\beta-2(\beta-\alpha)\cos^2(3\phi))} - \frac{2[(\beta+\alpha)^2 - (\beta-\alpha)^2]}{(2\beta-2(\beta-\alpha)\cos^2(3\phi))^2}]$; $\alpha = \frac{1}{2}\sqrt{1+2f_1}$, $\beta = \frac{1}{2}\sqrt{1+2g}$, (V(ϕ) = RE Trigonometric Pöschl- Teller II potential)	$9(\alpha + \beta + 1 + 2\ell)^2$	$\propto \frac{(1-z)^{\frac{\alpha}{2}+\frac{1}{4}}(1+z)^{\frac{\beta}{2}+\frac{1}{4}}}{(2\beta-2(\beta-\alpha)\xi)}$ $\times \hat{P}_{\ell+1}^{(\alpha,\beta)}(z)$; $z = (2\xi - 1) = \cos(6\phi)$.
(II) $V = V_C + V_{int} + V_{rat}$, $V_{int} = \frac{-f_1}{(\sqrt{6})r} [\frac{(x_1+x_2-2x_3)}{(x_1-x_2)^2} + c.p]$.	$\xi = (\frac{3}{\sqrt{6}})^{\frac{1}{r}} [(x_1 + x_2 - 2x_3)^{-1} + c.t]^{-1}$ or $\xi = -\frac{4}{(6\sqrt{6})r^3} (x_1 + x_2 - 2x_3) \times (x_2 + x_3 - 2x_1)(x_1 + x_3 - 2x_2)$; $\delta = 9; k_1 = 2(\beta + \alpha)$; $k_2 = (\beta + \alpha); k_3 = (\beta - \alpha)$; $k_4 = 2[(\beta + \alpha)^2 - (\beta - \alpha)^2]$.	$V_{Con}(\phi) = \frac{9g}{2} \operatorname{cosec}^2(3\phi)$ $- \frac{9f_1}{2} \operatorname{cosec}(3\phi) \cot(3\phi)$; $V_{rat}^{(2)}(\phi) = 9[\frac{2(\beta+\alpha)}{(\beta+\alpha)-(\beta-\alpha)\cos(3\phi)} - \frac{2[(\beta+\alpha)^2 - (\beta-\alpha)^2]}{[(\beta+\alpha)-(\beta-\alpha)\cos(3\phi)]^2}]$; $\alpha = [\frac{1}{2}(g + f_1 + \frac{1}{2})]^{\frac{1}{2}}$, $\beta = [\frac{1}{2}(g - f_1 + \frac{1}{2})]^{\frac{1}{2}}; f_1 < g + \frac{1}{2}$. (V(ϕ) = RE trigonometric Pöschl - Teller potential)	$9(\frac{\alpha+\beta+1}{2} + \ell)^2$	$\propto \frac{(1-z)^{\frac{\alpha}{2}+\frac{1}{4}}(1+z)^{\frac{\beta}{2}+\frac{1}{4}}}{((\beta+\alpha)-(\beta-\alpha)z)}$ $\times \hat{P}_{\ell+1}^{(\alpha,\beta)}(z)$, $z = -\xi = \cos(3\phi)$
(III) $V = V_H + V_W(g) + V_{int} + V_{rat}$, $V_{int} = \frac{-3\sqrt{3}}{2r^2} i f_1 [\frac{(x_1-x_2)}{(x_1+x_2-2x_3)} + c.p]$.	$\xi = \frac{1}{\sqrt{3}} (\frac{x_1-x_2}{x_1+x_2-2x_3} + c.p)$ $k_1 = \frac{4B}{A^2(A+3)^2} [A^2(A+3)^2 - B^2]$; $k_2 = B; k_3 = iA(A+3)$; $k_4 = \frac{2}{A^2(A+3)^2} [A^2(A+3)^2 - B^2]^2$; $\delta = -9$.	$V_{Con}(\phi) = A(A+B) \sec^2(3\phi)$ $- 2iB \tan(3\phi)$; $V_{rat}^{(2)}(\phi) = -\frac{9}{A^2(A+3)^2}$ $\times [\frac{4B(A^2(A+3)^2 - B^2)}{(B+iA(A+3)\tan(3\phi))} + \frac{2(A^2(A+3)^2 - B^2)^2}{(B+iA(A+3)\tan(3\phi))^2}]$; $A = \frac{3}{2}(\sqrt{1+2g} - 1); g > -\frac{1}{2}$, $B = \frac{9f_1}{4}$. ((V(ϕ) = RE PT symmetric trigonometric RM potential)	$(A+3-3\ell)^2 + \frac{B^2}{(A+3-3\ell)^2}$	$\propto \frac{(1-z)^{\frac{\alpha_\ell}{2}}(1+z)^{\frac{\beta_\ell}{2}}}{(B+iA(A+3)\xi)} y_\nu^{(A/3,B/3)}(z)$; $y_\nu^{(A/3,B/3)}(z) = \frac{2(\ell+\alpha_\ell)(\ell+\beta_\ell)}{2\ell+\alpha_\ell+\beta_\ell}$ $\times q_1^{(A/3,B/3)}(z) P_{\ell-1}^{(\alpha_\ell,\beta_\ell)}(z) - \frac{2(1+\alpha_1)(1+\beta_1)}{(2+\alpha_1+\beta_1)} P_\ell^{(\alpha_\ell,\beta_\ell)}(z)$; $\alpha_\ell = \frac{A}{3} + 1 - \ell + \frac{B/9}{\frac{A}{3}+1-\ell}$; $\beta_\ell = \frac{A}{3} + 1 - \ell - \frac{B/9}{\frac{A}{3}+1-\ell}$; $q_1^{(A/3,B/3)}(z) = P_1^{(\alpha_1,\beta_1)}(z)$; $z = i\xi = i \tan(3\phi)$.

Table 1: Rationally extended (RE) three body problems and their equivalent RE real and PT symmetric complex potentials. The ϕ dependent eigenfunctions ($\Phi_\ell(\phi)$) of these potentials are written in terms of X_1 EOPs ($\hat{P}_{\ell+1}^{(\alpha,\beta)}(z)$) or in terms of some other type of polynomials $y_\nu^{(A/3,B/3)}(z)$. The terms V_C and V_H are same as given by Eqs. (1) and (2) in the text. The complete wave functions and the corresponding energy eigenvalues are given by Eqs. (19) and (26).

Forms of potential ($V(m)$)	Forms of χ or z	$V(p, \phi) = V_{Con}(\phi) + V_{rat}^{(2)}(p, \phi)$	$\Phi_{\ell,p}(\phi)$
(I) $V(p) = V_H + V_W(g) + V_{int} + V_{rat}(p)$	$\chi = z^2 - 1; \quad z = 2\xi - 1$	$V_{rat}^{(2)}(p, \phi) = 36 \left[-2p(\alpha - \beta - p + 1) - (\alpha - \beta - p + 1)(\alpha + \beta + (\alpha - \beta + 1)z) \right. \\ \times \frac{P_{p-1}^{(-\alpha, \beta)}(z)}{P_p^{(-\alpha-1, \beta-1)}(z)} + \frac{(\alpha - \beta - p + 1)^2 \chi}{2} \\ \left. \times \left(\frac{P_{p-1}^{(-\alpha, \beta)}(z)}{P_p^{(-\alpha-1, \beta-1)}(z)} \right)^2 \right]$ <p>(RE Trigonometric Pöschl-Teller II potential)</p>	$\propto \frac{(1-z)^{\frac{\alpha}{2} + \frac{1}{4}} (1+z)^{\frac{\beta}{2} + \frac{1}{4}}}{P_{p-1}^{(-\alpha-1, \beta-1)}(z)} \hat{P}_{\ell+p}^{(\alpha, \beta)}(z).$
(III) $V(p) = V_C + V_{int} + V_{rat}(p)$	$\chi = z^2 - 1; \quad z = -\xi$	$V_{rat}^{(2)}(p, \phi) = 9 \left[-2p(\alpha - \beta - p + 1) - (\alpha - \beta - p + 1)(\alpha + \beta + (\alpha - \beta + 1)z) \right. \\ \times \frac{P_{p-1}^{(-\alpha, \beta)}(z)}{P_p^{(-\alpha-1, \beta-1)}(z)} + \frac{(\alpha - \beta - p + 1)^2 \chi}{2} \\ \left. \times \left(\frac{P_{p-1}^{(-\alpha, \beta)}(z)}{P_p^{(-\alpha-1, \beta-1)}(z)} \right)^2 \right]$ <p>(RE Trigonometric Pöschl-Teller potential)</p>	$\propto \frac{(1-z)^{\frac{\alpha}{2} + \frac{1}{4}} (1+z)^{\frac{\beta}{2} + \frac{1}{4}}}{P_{p-1}^{(-\alpha-1, \beta-1)}(z)} \hat{P}_{\ell+p}^{(\alpha, \beta)}(z)$
(III) $V(p) = V_H + V_W(g) + V_{int} + V_{rat}(p)$.	$z = i\xi$	$V_{rat}^{(2)}(p, \phi) = -18(1 + \xi^2) \left[2i\xi \frac{\dot{q}_p^{(A/3, B/3)}(z)}{q_p^{(A/3, B/3)}(z)} - (1 + \xi)^2 \left(\frac{\ddot{q}_p^{(A/3, B/3)}(z)}{q_p^{(A/3, B/3)}(z)} - \left(\frac{\dot{q}_p^{(A/3, B/3)}(z)}{q_p^{(A/3, B/3)}(z)} \right)^2 \right) - p \right];$ <p>(RE PT symmetric trigonometric RM potential)</p>	$\propto \frac{(1-z)^{\frac{\alpha_\ell}{2}} (1+z)^{\frac{\beta_\ell}{2}}}{q_p^{(A/3, B/3)}(z)} y_{\nu,p}^{(A/3, B/3)}(z);$ $y_{\nu,p}^{(A/3, B/3)}(z) = \frac{2(\ell + \alpha_\ell)(\ell + \beta_\ell)}{2\ell + \alpha_\ell + \beta_\ell} q_p^{(A/3, B/3)}(z)$ $\times P_{\ell-1}^{(\alpha_\ell, \beta_\ell)}(z) - \frac{2(p + \alpha_p)(p + \beta_p)}{(2p + \alpha_p + \beta_p)}$ $\times q_{p-1}^{(A/3-1, B/3)}(z) P_\ell^{(\alpha_\ell, \beta_\ell)}(z).$

Table 2: RE three body problems and their ϕ dependent eigenfunctions ($\Phi_\ell(\phi)$) for any values of p are written in terms of X_p EOPs ($\hat{P}_{\ell+p}^{(\alpha, \beta)}(z)$) or in terms of some other type of polynomials $y_{\nu,p}^{(A, B)}(z)$. The λ_ℓ will be same as that of the $m = 1$ case. The complete wave functions and the corresponding energy eigenvalues are given by Eqs. (19) and (26).